Flight Dynamics Model for the Real-Time Flight Simulation

Author: Marek M. Cel
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Notation

a	[m] ellipsoid equatorial radius
b	[m] ellipsoid polar radius
$\hat{c}$	[m] mean aerodynamic chord
c	[N/(m/s)] damping coefficient
c_S	[m/s] speed of sound
C_D	[-] drag coefficient
C_l	[-] rolling moment coefficient
C_L	[-] lift coefficient
C_m	[-] pitching moment coefficient
C_n	[-] yawing moment coefficient
$C_P = \dfrac{P}{\rho n^3 D^5}$	[-] power coefficient
$C_T = \dfrac{T}{\rho n^2 D^4}$	[-] thrust coefficient
C_Y	[-] side force coefficient
$C_{\Delta P}$	[-] engine power losses coefficient
D	[N] drag
D	[m] propeller diameter
g	[m/s^2] gravity acceleration
h	[m] altitude
$\vec{H} = [H_X, H_Y, H_Z]^T$	[kg·m^2/s] angular momentum vector
i	[rad] incidence angle
$\boldsymbol{I}$	[kg·m^2] inertia matrix
J	[-] propeller advance ratio
k	[N/m] spring constant
L	[N] lift
m	[kg] mass
$\boldsymbol{M}$	[kg],[kg·m],[kg·m^2] generalized inertia matrix
n	[rpm] engine revolutions per minute
n	[rev./s] propeller revolution speed

p	[Pa] pressure
P	[W] power
$\vec{P} = [P_X, P_Y, P_Z]^T$	[kg·m/s] momentum vector
Q	[N·m] torque
$\vec{Q} = [L, M, N]^T$	[N·m] moment of force vector
$\vec{r} = [x, y, z]^T$	[m] coordinates vector
R	[J/(kmol·K)] universal gas constant
$\vec{R} = [X, Y, Z]^T$	[N] force vector
$\boldsymbol{s} = [u, v, w, p, q, r]^T$	[m/s],[rad/s] aircraft state vector
S	[m^2] wing area
S	[K] Sutherland constant
$\vec{S} = [S_X, S_Y, S_Z]^T$	[kg·m] first moment of mass
T	[K] temperature
T	[N] thrust
V	[m/s] velocity
$\vec{V} = [u, v, w]^T$	[m/s] velocity vector
$\boldsymbol{x} = [x, y, z, e_0, e_x, e_y, e_z]^T$	[m],[-] aircraft coordinates vector
α	[rad] angle of attack
β	[rad] angle of sideslip
$\hat{\delta}$	[-] normalized controls position
θ	[rad] pitch angle
λ	[rad] geographic longitude
μ	[-] friction coefficient
μ	[Pa·s] dynamic viscosity
ν	[m^2/s] kinematic viscosity
ρ	[kg/m^3] air density
φ	[rad] geographic latitude
ϕ	[rad] roll angle
ψ	[rad] yaw angle
$\vec{\omega} = [p, q, r]^T$	[rad/s] angular velocity vector
$\dfrac{\partial \epsilon}{\partial \alpha}$	[-] horizontal stabilizer downwash angle derivative with respect to the aircraft angle of attack

Indices:

a	Aerodynamic Axis System
A	aerodynamics
AC	aerodynamic center
b	Body Axis System

CM	Center of Mass
e	Earth-fixed Axis System
g	Gravity (North-East-Down) Axis System
h	horizontal stabilizer
LG	landing gear
M	mass
MAP	manifold absolute pressure
N	normal
O	coordinate system's origin
P	propulsion
s	Stability Axis System
T	tangent
v	vertical stabilizer
X	x-axis component
Y	y-axis component
Z	z-axis component

Chapter 1

Introduction

1.1 Conventions

Flight Dynamics Model uses International System of Units (SI) for all internal computations. It is clearly specified if other units are used.

All rotations and rotation related operations are considered to be a passive (alias) rotations.

1.2 Coordinates Systems

1.2.1 Body Axis System

Body Axis System is body-centered, body-fixed coordinate system, with the x-axis positive forwards, the y-axis positive right and the z-axis positive downwards.

1.2.2 Stability Axis System

Origin of the Stability Axis System is coincident with the origin of the Body Axis System, the x-axis is directed along air freestream velocity vector projected onto the XZ plane of the Body Axis System, the y-axis is coincident with the the y-axis

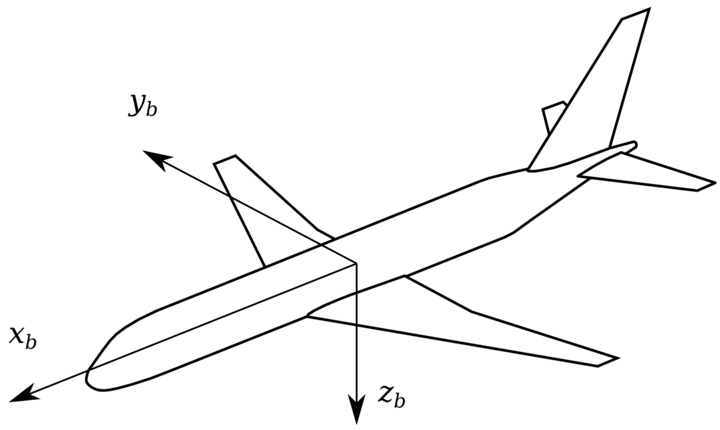

Figure 1.1: Body Axis System

of the Body Axis System, and the z-axis completes a right-handed coordinate system.

1.2.3 Aerodynamic Axis System

Origin of the Aerodynamic Axis System is coincident with the origin of the Body Axis System, the x-axis is directed along air freestream velocity vector, the z-axis lies in the aircraft plane of symmetry pointing upwards, and the y-axis completes a right-handed coordinate system.

1.2.4 Gravity Axis System

There are basically two conventions of Gravity Axis Systems used for the purpose of flight dynamics East-North-Up and North-East-Down.

It is convenient to use North-East-Down axes system together with the Body Axis System and Bryant angles (Euler angles in z-y-x convention) as those angles in NED frame become aircraft heading, pitch and roll.

Considering all this, Gravity Earth Axis System is a coordinate system, with the x-axis positive North, the y-axis positive East and z-axis positive downwards.

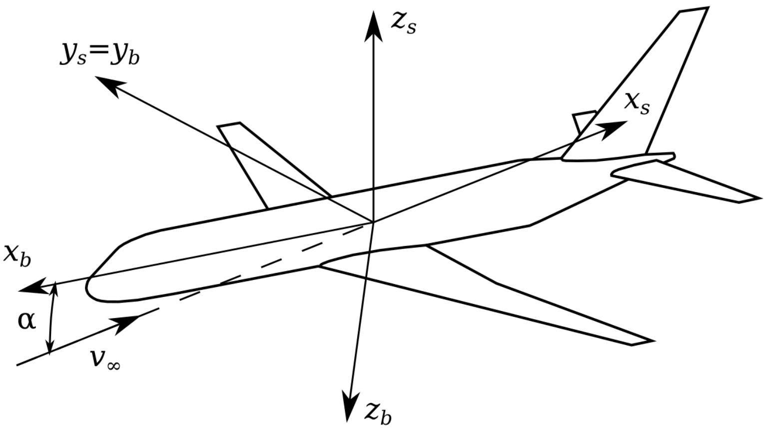

Figure 1.2: Stability Axis System

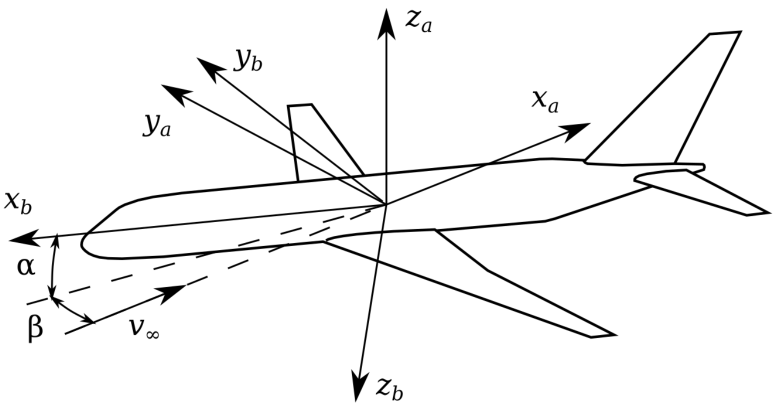

Figure 1.3: Aerodynamic Axis System

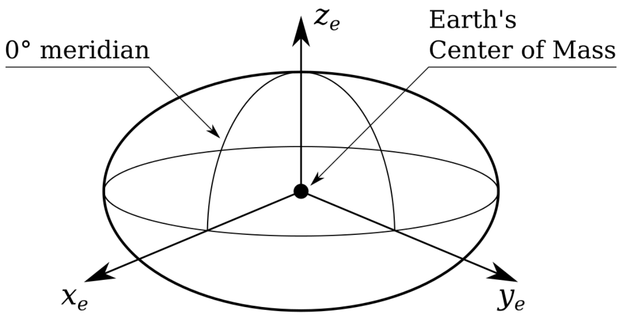

Figure 1.4: World Geodetic System 1984

1.2.5 Earth-fixed Axis System

For any further considerations World Geodetic System 1984 as described in [1] is used as the Earth-fixed Axis System.

1.3 Aircraft Attitude

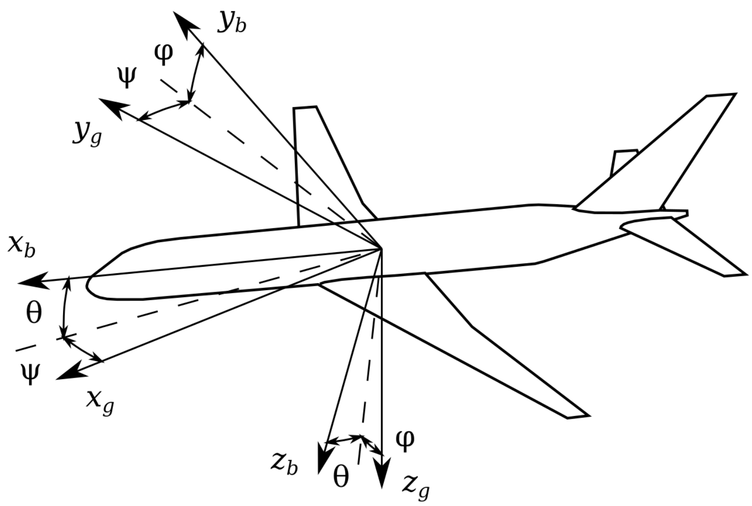

Figure 1.5: Tait–Bryan angles

Aircraft attitude is defined either by a quaternion or by quasi-Euler Tait-Bryan ψ-θ-ϕ angles in z-y-x convention. It is convenient to use such a convention, as this angles becomes aircraft roll, pitch and heading when expressed in North-East-Down coordinate system.

1.4 Transformations Between Coordinates Systems

1.4.1 Rotation Matrices

ransformation to the coordinate system rotated by Tait-Bryan ψ-θ-ϕ angles can be performed using rotation matrix, which is given by the following relations. [2, 3]

$$\boldsymbol{T}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{1.1}$$

$$\boldsymbol{T}(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{1.2}$$

$$\boldsymbol{T}(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.3}$$

$$\boldsymbol{T}(\phi,\theta,\psi) = \boldsymbol{T}(\phi)\,\boldsymbol{T}(\theta)\,\boldsymbol{T}(\psi) = \\ \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \cos\theta\sin\phi \\ \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta & \cos\phi\sin\theta\sin\psi - \cos\psi\sin\phi & \cos\phi\cos\theta \end{bmatrix} \tag{1.4}$$

1.4.2 Geographic Coordinates

Conversion from Geographic to Cartesian Coordinates

Procedure of conversion from geographic coordinates to the Cartesian coordinates system is given as follows.

$$e = \frac{1}{a}\sqrt{a^2 - b^2} \tag{1.5}$$

$$\chi = \sqrt{1 - e^2 \sin^2 \varphi} \tag{1.6}$$

$$x_e = \left(\frac{a}{\chi} + h\right) \cos\varphi \cos\lambda \tag{1.7}$$

$$y_e = \left(\frac{a}{\chi} + h\right) \cos\varphi \sin\lambda \tag{1.8}$$

$$z_e = \left(a\frac{1 - e^2}{\chi} + h\right) \sin\varphi \tag{1.9}$$

Conversion from Cartesian to Geographic Coordinates

Reverse conversion is given as follows. [4]

$$r = \sqrt{x_e^2 + y_e^2} \tag{1.10}$$

$$E^2 = a^2 - b^2 \tag{1.11}$$

$$e'^2 = \frac{a^2 - b^2}{b^2} \tag{1.12}$$

$$F = 54b^2 z_e^2 \tag{1.13}$$

$$G = r^2 + \left(1 - e^2\right) z_e^2 - e^2 E^2 \tag{1.14}$$

$$C = \frac{e^4 F r^2}{G^3} \tag{1.15}$$

$$S = \sqrt[3]{1 + C + \sqrt{C^2 + 2C}} \tag{1.16}$$

$$P_0 = S + \frac{1}{S} + 1 \tag{1.17}$$

$$P = \frac{F}{3P_0^2 G^2} \tag{1.18}$$

$$Q = \sqrt{1 + 2e^4 P} \tag{1.19}$$

$$r_0 = \frac{-\left(Pe^2 r\right)}{1+Q} + \sqrt{\frac{1}{2}a^2\left(1+\frac{1}{Q}\right) - \frac{P\left(1-e^2\right)z_e^2}{Q+Q^2} - \frac{1}{2}Pr^2} \tag{1.20}$$

$$U_0 = r - e^2 r_0 \tag{1.21}$$

$$U = \sqrt{U_0^2 + z_e^2} \tag{1.22}$$

$$V = \sqrt{U_0^2 + \left(1-e^2\right)z_e^2} \tag{1.23}$$

$$Z_0 = \frac{b^2 z_e}{aV} \tag{1.24}$$

Geographic coordinates are given as.

$$h = U\left(1 - \frac{b^2}{aV}\right) \tag{1.25}$$

$$\varphi = \arctan\left(\frac{z_e + e'^2 Z_0}{r}\right) \tag{1.26}$$

$$\lambda = \arctan\left(\frac{y_e}{z_e}\right) \tag{1.27}$$

Chapter 2

Flight Dynamics Model

2.1 Assumptions

Following assumptions are made:

— forces and moments acting on the aircraft are considered to be quasi-steady,

— aircraft is considered to be a rigid body,

— mass and moments of inertia depend only on variable masses (fuel, payload, etc.).

2.2 Equations of Motion

2.2.1 Dynamic Equations

Dynamic equations of motion are derived in Body Axis System for a rigid aircraft using conservation of momentum and angular momentum principles which are given by the following formulas. [5, 6, 7]

$$\frac{d\vec{P}_b}{dt} = \sum_j \vec{R}_{j,b} \tag{2.1}$$

$$\frac{d\vec{H}_{O,b}}{dt} + \vec{V}_{O,b} \times \vec{P}_b = \sum_j \vec{Q}_{O,j,b} \tag{2.2}$$

Where:

$$\sum_j \vec{R}_{j,b} = \vec{R}_{A,b} + \vec{R}_{M,b} + \vec{R}_{LG,b} + \vec{R}_{P,b} \tag{2.3}$$

$$\sum_j \vec{Q}_{O,j,b} = \vec{Q}_{O,A,b} + \vec{Q}_{O,M,b} + \vec{Q}_{O,LG,b} + \vec{Q}_{O,P,b} \tag{2.4}$$

Momentum and angular momentum are: [6, 7]

$$\vec{P}_b = m\vec{V}_{CM,b} \tag{2.5}$$

$$\vec{H}_{O,b} = \boldsymbol{I}_{O,b}\vec{\omega}_b + m\left(\vec{r}_{CM,b} \times \vec{V}_{O,b}\right) \tag{2.6}$$

Center of mass velocity is:

$$\vec{V}_{CM,b} = \vec{V}_{O,b} + \vec{\omega}_b \times \vec{r}_{CM,b} \tag{2.7}$$

Substituting equation (2.7) into equations (2.5) and (2.6) gives:

$$\vec{P}_b = m\vec{V}_{O,b} + \vec{\omega}_b \times \vec{S}_b \tag{2.8}$$

$$\vec{H}_{O,b} = \boldsymbol{I}_b\vec{\omega}_b + \vec{S}_b \times \vec{V}_{O,b} \tag{2.9}$$

Where:

$$\vec{S}_b = [S_X, S_Y, S_Z]^T = m\vec{r}_{CM,b} \tag{2.10}$$

Derivatives of momentum and angular momentum in rotating reference frame are: [5, 6, 7]

$$\frac{d\vec{P}_b}{dt} = \frac{\delta\vec{P}_b}{\delta t} + \vec{\omega}_b \times \vec{P}_b \tag{2.11}$$

$$\frac{d\vec{H}_{O,b}}{dt} = \frac{\delta\vec{H}_{O,b}}{\delta t} + \vec{\omega}_b \times \vec{H}_{O,b} \tag{2.12}$$

Substituting equations (2.11) and (2.12) into (2.1) and (2.2) gives:

$$\frac{\delta \vec{P}_b}{\delta t} = \sum_j \vec{R}_{j,b} - \vec{\omega}_b \times \vec{P}_b \tag{2.13}$$

$$\frac{\delta \vec{H}_{O,b}}{\delta t} = \sum_j \vec{Q}_{O,j,b} - \vec{V}_{O,b} \times \vec{P}_b - \vec{\omega}_b \times \vec{H}_{O,b} \tag{2.14}$$

Differentiating equations (2.8) and (2.9) gives:

$$\frac{\delta \vec{P}_b}{\delta t} = m\frac{\delta \vec{V}_{O,b}}{\delta t} + \frac{\delta \vec{\omega}_b}{\delta t} \times \vec{S}_b \tag{2.15}$$

$$\frac{\delta \vec{H}_{O,b}}{\delta t} = \boldsymbol{I}_b \frac{\delta \vec{\omega}_b}{\delta t} + \vec{S}_b \times \frac{\delta \vec{V}_{O,b}}{\delta t} \tag{2.16}$$

Substituting equations (2.15) and (2.16) into (2.13) and (2.14) gives:

$$m\frac{\delta \vec{V}_{O,b}}{\delta t} + \frac{\delta \vec{\omega}_b}{\delta t} \times \vec{S}_b = \sum_j \vec{R}_{j,b} - \vec{\omega}_b \times \vec{P}_b \tag{2.17}$$

$$\boldsymbol{I}_b \frac{\delta \vec{\omega}_b}{\delta t} + \vec{S}_b \times \frac{\delta \vec{V}_{O,b}}{\delta t} = \sum_j \vec{Q}_{O,j,b} - \vec{V}_{O,b} \times \vec{P}_b - \vec{\omega}_b \times \vec{H}_{O,b} \tag{2.18}$$

Representing vector cross product as matrix-vector multiplication equations (2.17) and (2.18) can be written as:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} 0 & S_Z & -S_Y \\ -S_Z & 0 & S_X \\ S_Y & -S_X & 0 \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \sum_j \vec{R}_{j,b} - \vec{\omega}_b \times \vec{P}_b \tag{2.19}$$

$$\begin{aligned} \begin{bmatrix} I_X & -I_{XY} & -I_{XZ} \\ -I_{XY} & I_Y & -I_{YZ} \\ -I_{XZ} & -I_{YZ} & I_Z \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} 0 & -S_Z & S_Y \\ S_Z & 0 & -S_X \\ -S_Y & S_X & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \\ = \sum_j \vec{Q}_{O,j,b} - \vec{V}_{O,b} \times \vec{P}_b - \vec{\omega}_b \times \vec{H}_{O,b} \end{aligned} \tag{2.20}$$

Combined equations (2.19) and (2.20) can be written as follows. [3]

$$\boldsymbol{M}\dot{\boldsymbol{s}} = \boldsymbol{R} \tag{2.21}$$

Where:

$$\dot{\boldsymbol{s}} = [\dot{u}, \dot{v}, \dot{w}, \dot{p}, \dot{q}, \dot{r}]^T \tag{2.22}$$

$$\boldsymbol{R} = \begin{bmatrix} \sum_j \vec{R}_{j,b} - \vec{\omega}_b \times \vec{P}_b \\ \sum_j \vec{Q}_{O,j,b} - \vec{V}_{O,b} \times \vec{P}_b - \vec{\omega}_b \times \vec{H}_{O,b} \end{bmatrix} \tag{2.23}$$

$$\boldsymbol{M} = \begin{bmatrix} m & 0 & 0 & 0 & S_Z & -S_Y \\ 0 & m & 0 & -S_Z & 0 & S_X \\ 0 & 0 & m & S_Y & -S_X & 0 \\ 0 & -S_Z & S_Y & I_X & -I_{XY} & -I_{XZ} \\ S_Z & 0 & -S_X & -I_{XY} & I_Y & -I_{YZ} \\ -S_Y & S_X & 0 & -I_{XZ} & -I_{YZ} & I_Z \end{bmatrix} \tag{2.24}$$

For the purpose of numerical simulation equation (2.21) can be written in the following form, which is easy to solve with Gaussian methods.

$$\dot{\boldsymbol{s}} = \boldsymbol{M}^{-1}\boldsymbol{R} \tag{2.25}$$

2.2.2 Kinematic Equations

Time Derivatives

Position vector derivative is given as follows. [8]

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \cos\phi\sin\theta\sin\psi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{2.26}$$

Tait-Bryan angles derivatives are given as follows. [3, 8]

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{2.27}$$

There are singularities in equation (2.27) for value of $\theta = \pm 90°$. One method of solving this problem is to use quaternions instead of Tait-Bryan angles to describe aircraft attitude.

Quaternions

Quaternion time derivative is given as follows. [3, 9]

$$\begin{bmatrix} \dot{e}_0 \\ \dot{e}_X \\ \dot{e}_Y \\ \dot{e}_Z \end{bmatrix} = \begin{bmatrix} 1 & p & q & r \\ -p & 0 & -r & q \\ -q & r & 0 & -p \\ -r & -q & p & 0 \end{bmatrix} \begin{bmatrix} e_0 \\ e_X \\ e_Y \\ e_Z \end{bmatrix} \tag{2.28}$$

2.3 Numerical Integration

State vector $\boldsymbol{s}$ can be calculated by solving initial value problem given by the following expression.

$$\boldsymbol{s}\left(t_0 + \Delta t\right) = \boldsymbol{s}\left(t_0\right) + \int_{t_0}^{t_0+\Delta t} \dot{\boldsymbol{s}} dt \tag{2.29}$$

State vector derivative $\dot{\boldsymbol{s}}$ can be calculated using formula (2.25).

Aircraft position and attitude can be calculated by solving initial value problem given as follows.

$$\boldsymbol{x}\left(t_0 + \Delta t\right) = \boldsymbol{x}\left(t_0\right) + \int_{t_0}^{t_0+\Delta t} \dot{\boldsymbol{x}} dt \tag{2.30}$$

Where:

$$\boldsymbol{x} = \left[x, y, z, e_0, e_X, e_Y, e_Z\right]^T \tag{2.31}$$

Coordinates vector derivative $\dot{\boldsymbol{x}}$ can be calculated using formulas (2.26) and (2.28).

Initial value problems, given by the (2.29) and (2.30) expressions, can be solved using Runge-Kutta 4th-order method which is given as follows. [10, 11, 12]

$$y\left(t_0 + \Delta t\right) \approx y\left(t_0\right) + \frac{1}{6}\Delta t\left(k_1 + 2k_2 + 2k_3 + k_4\right) \tag{2.32}$$

Where:

$$k_1 = f\left(t_n, y_n\right) \tag{2.33}$$

$$k_2 = f\left(t_n + \frac{1}{2}\Delta t, y_n + \frac{1}{2}\Delta t k_1\right) \tag{2.34}$$

$$k_3 = f\left(t_n + \frac{1}{2}\Delta t, y_n + \frac{1}{2}\Delta t k_2\right) \tag{2.35}$$

$$k_4 = f\left(t_n + \Delta t, y_n + \Delta t k_3\right) \tag{2.36}$$

$$\tag{2.37}$$

Chapter 3

Environment

3.1 Atmosphere

US Standard Atmosphere 1976 is used to calculate air temperature, pressure, density, viscosity and speed of sound depending on altitude.

Mean molecular weight is given as follows:

$$M_0 = \frac{\sum_j M_j F_j}{\sum_j F_j} = 28.9645 \tag{3.1}$$

Temperature is given by the following formula: [13]

$$T\left(h\right) = T_j + \left(\frac{dT}{dh}\right)_j \left(h - h_j\right) \tag{3.2}$$

Pressure is given as follows: [13]

$$p\left(h\right) = p_j \left(\frac{T_j}{T\left(h\right)}\right)^{\frac{gM_0}{R\left(\frac{dT}{dh}\right)_j}} \text{ for } \left(\frac{dT}{dh}\right)_j \neq 0 \tag{3.3}$$

$$p\left(h\right) = p_j e^{\frac{gM_0\left(h-h_j\right)}{RT_j}} \text{ for } \left(\frac{dT}{dh}\right)_j = 0 \tag{3.4}$$

Density is expressed by the following formula: [13]

$$\rho\left(h\right) = \frac{p\left(h\right) M_0}{RT\left(h\right)} \tag{3.5}$$

Speed of sound is given as follows: [13]

$$c_S\left(h\right) = \sqrt{\frac{\gamma RT\left(h\right)}{M_0}} \tag{3.6}$$

Dynamic viscosity is given by the formula: [13]

$$\mu\left(h\right) = \frac{1.458 \cdot 10^{-6} \sqrt{\left[T\left(h\right)\right]^3}}{T\left(h\right) + S} \tag{3.7}$$

Kinetic viscosity is given as follows: [13]

$$\nu\left(h\right) = \frac{\mu\left(h\right)}{\rho\left(h\right)} \tag{3.8}$$

Altitude h_j [m]	**Temperature gradient** $\left(\frac{dT}{dh}\right)_j$ [K/m]	**Temperature** T_j [K]	**Pressure** p_j [Pa]
0	-6.5×10^{-3}	288.15	101 325.0
11 000	0.0	216.65	22 632.0
20 000	1.0×10^{-3}	216.65	5474.8
32 000	2.8×10^{-3}	228.65	868.01
47 000	0.0	270.65	110.9
51 000	-2.8×10^{-3}	270.65	66.938
71 000	-2.0×10^{-3}	214.65	3.9564

Table 3.1: Reference levels [13]

Gas species	**Molecular weight** [kg/kmol]	**Fractional volume** [-]
Nitrogen	28.0134	0.780 84
Oxygen	31.9988	0.209 476
Argon	39.948	0.009 34
Carbon Dioxide	44.009 95	0.000 314
Neon	20.183	0.000 018 18
Helium	4.0026	0.000 005 24
Krypton	83.8	0.000 001 14
Xenon	131.3	0.000 000 087
Methane	16.043 03	0.000 002
Hydrogen	2.015 94	0.000 000 5

Table 3.2: Molecular weights and fractional volume composition of S/L dry air [13]

Chapter 4

Aerodynamics

Aerodynamic forces are calculated in Aerodynamic Axis System, while moments are calculated in Stability Axis System. Rotation matrix from Stability Axis System to Body Axis System can be calculated using formula (4.1). Rotation matrix from Aerodynamic Axis System to Body Axis System can be calculated using following formulas:

$$\boldsymbol{T}(\alpha) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} = \begin{bmatrix} -\cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & -\cos\alpha \end{bmatrix} \tag{4.1}$$

$$\boldsymbol{T}(\beta) = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2}$$

$$\boldsymbol{T}(\alpha, \beta) = \boldsymbol{T}(\alpha)\,\boldsymbol{T}(\beta) = \begin{bmatrix} -\cos\alpha\cos\beta & -\cos\alpha\sin\beta & \sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ -\sin\alpha\cos\beta & -\sin\alpha\sin\beta & -\cos\alpha \end{bmatrix} \tag{4.3}$$

Considering a no-wind conditions angle of attack and angle of sideslip (positive when the aircraft velocity component along the transverse axis is positive [14]) are given as follows:

$$\alpha = \arctan\left(\frac{w}{\sqrt{u^2 + v^2}}\right) \tag{4.4}$$

$$\beta = \arcsin\left(\frac{v}{V}\right) \tag{4.5}$$

4.1 Tail-off Aircraft

Tail-off aircraft aerodynamics model is intended to be used in application, e.g. fixed-wing aircrafts, where asymmetric aerodynamic effects, such as autorotation spin or roll damping, are significant.

Forces and moments are calculated for each half-wing to consider asymmetric effects. Half wing aerodynamic center velocity vector used to calculate angle of attack, angle of sideslip as well as forces and moments is given as follows:

$$\vec{V}_{AC} = \vec{V}_O + \vec{\Omega} \times \vec{r}_{AC} \tag{4.6}$$

Forces and moments generated by the half-wing are given as follows: [9]

$$\vec{F}_a = \left[F_{X,a}, F_{Y,a}, F_{Z,a}\right]^T \tag{4.7}$$
$$\vec{M}_s = \left[M_{X,s}, M_{Y,s}, M_{Z,s}\right]^T \tag{4.8}$$

Where:

$$F_{X,a} = \frac{1}{2}\rho V^2 S C_D \tag{4.9}$$
$$F_{Y,a} = \frac{1}{2}\rho V^2 S C_Y \tag{4.10}$$
$$F_{Z,a} = \frac{1}{2}\rho V^2 S C_L \tag{4.11}$$
$$M_{X,s} = \frac{1}{2}\rho V^2 S \hat{c} C_l \tag{4.12}$$
$$M_{Y,s} = \frac{1}{2}\rho V^2 S \hat{c} C_m \tag{4.13}$$
$$M_{Z,s} = \frac{1}{2}\rho V^2 S \hat{c} C_n \tag{4.14}$$

Forces and moments generated by the half-wing expressed in Body Axis System are given by the following formulas:

$$\vec{F}_b = \boldsymbol{T}\left(\alpha, \beta\right) \vec{F}_a \tag{4.15}$$
$$\vec{M}_b = \boldsymbol{T}\left(\alpha\right) \vec{M}_s + \vec{r}_{AC,b} \times \vec{F}_b \tag{4.16}$$

4.2 Fuselage

Fuselage aerodynamics model is intended to be used in application where asymmetric aerodynamic effects can be neglected, e.g. to model helicopter fuselage. It is very much like, described above, tail-off aircraft model. The main difference is that calculations are performed for whole fuselage unlike the tail-off aircraft where calculations are performed for each half-wing.

4.3 Stabilizers

Velocity vector used to calculate stabilizer angle of attack, angle of sideslip as well as forces and moments is calculated using expression (4.6).

Horizontal stabilizer angle of attack is modified due to incidence angle and downwash angle, what can be expressed as follows. [15]

$$\Delta\alpha_h = i_h + \frac{\partial\epsilon}{\partial\alpha}\alpha \tag{4.17}$$

Forces generated by stabilizers are calculated using formulas (4.9), (4.10) and (4.11).

Formula (4.15) can be used to calculate stabilizer generated forces expressed in Body Axis System.

It is assumed that horizontal stabilizer generates only drag and lift, while vertical stabilizer generates only drag and side force. Moments generated by stabilizers comes only from force acting on arm, other moments are neglected.

$$\vec{M}_b = \vec{r}_{AC,b} \times \vec{F}_b \tag{4.18}$$

Chapter 5

Landing Gear

5.1 Contact Point

Landing gear contact point is considered to be an intersection of the ground plane and the line segment with the beginning at the strut attachment point and the end at the tire bottom.

Intersection of a line segment and a plane can be calculated using following expression: [16]

$$u = \frac{\vec{n} \cdot (\vec{r}_p - \vec{r}_b)}{\vec{n} \cdot (\vec{r}_e - \vec{r}_b)} \tag{5.1}$$

Where:

$\vec{n}$ — unit vector normal to the plane

$\vec{r}_b$ — position vector of the line segment beginning

$\vec{r}_e$ — position vector of the line segment end

$\vec{p}_b$ — position vector of any point on the plane

u — normalized coordinate of intersection point along line segment

If $0 \leq u \leq 1$ then intersection point is within line segment and its coordinates are given by the following formula:

$$\vec{r} = \vec{r}_b = u \left(\vec{r}_e - \vec{r}_r\right) \tag{5.2}$$

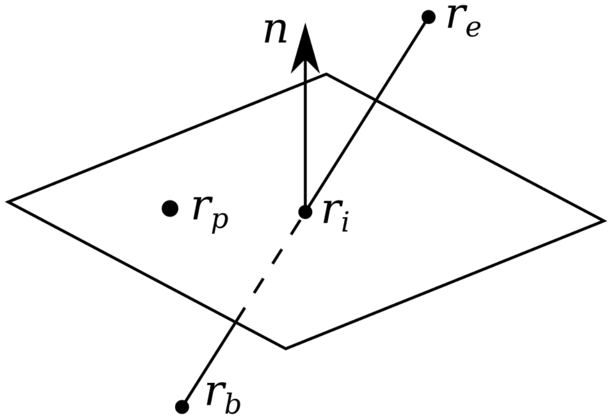

Figure 5.1: Segment-plane intersection

If denominator of expression (5.1) is zero then the line segment is parallel to the plane. If both numerator and denominator are zero then the line segment lies on the plane.

5.2 Forces and Moments

Forces generated by the landing gear can be divided into:

— normal to the ground plane forces due to struts and tires deflection,

— tangent to the ground plane forces due to friction between tires and the ground.

Normal forces are the sum of forces due to spring and damper while tangent force are caused by static or kinetic friction and optional rolling friction and are given as follows: [17]

$$F_N = kx + c\dot{x} \tag{5.3}$$
$$F_T = \mu F_N \tag{5.4}$$

Surface	**Static friction coefficient**	**Kinetic friction coefficient**
Concrete (dry)	0.8 - 1.0	0.7 - 0.8
Concrete (wet)	0.6 - 0.8	0.5 - 0.6
Tarmac (dry)	0.7 - 0.8	0.6 - 0.7
Tarmac (wet)	0.4 - 0.5	0.3 - 0.4
Dirt (dry)	0.5 - 0.6	0.2 - 0.3
Dirt (wet)	0.3 - 0.4	0.2 - 0.3
Snow	0.1 - 0.4	0.2 - 0.3
Ice	0.05 - 0.15	0.05 - 0.10

Table 5.1: Static and kinetic friction coefficients [18]

Surface	**Rolling friction coefficient**
Tarmac	0.010 - 0.012
Concrete	0.012 - 0.015
Dirt	0.030 - 0.140

Table 5.2: Static and kinetic friction coefficients [18]

Chapter 6

Mass and Inertia

6.1 Empty Aircraft Moments of Inertia

Aircraft is divided into structure groups which mass is estimated. This groups are assumed to be homogeneous rigid body with simple shape which allows to calculate its moment of inertia using an exact closed-form expression, given e.g. in [19].

Steiner's theorem, given by the following expression, is used to express aircraft structure groups inertia tensor in Body Axis System. [5, 17]

$$\boldsymbol{I}_b = \boldsymbol{I}_0 + m \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix} \tag{6.1}$$

Sum of all aircraft structure groups inertia tensors gives empty aircraft inertia tensor:

$$\boldsymbol{I}_b = \sum_j \boldsymbol{I}_{j,b} \tag{6.2}$$

6.2 Variable Masses

All variable masses, crew, fuel, payload, etc., are considered to be point masses. Point mass inertia tensor can be calculated using formula (6.1), where $\boldsymbol{I}_b = 0$. This tensors are then added to the empty aircraft inertia tensor giving total aircraft inertia tensor.

Aircraft total first moment of mass is given as follows:

$$\vec{S}_b = \sum_j m_j \vec{r}_{CM,j,b} \tag{6.3}$$

Position of aircraft center of mass including variable masses is then given by following formula:

$$\vec{r}_{CM,b} = \frac{\vec{S}_b}{\sum_j m_j} \tag{6.4}$$

Chapter 7

Propulsion

7.1 Piston Engine

Piston engine manifold absolute pressure due to engine revolution speed and normalized throttle position is approximated by the following formula.

$$p_{MAP} = p\left(h\right) + \left(4 \cdot 10^5 \hat{\delta}_{throttle} - 4.05 \cdot 10^5\right) \frac{n}{n_{max}} \tag{7.1}$$

Fuel to air ratio is approximated by the following expression: [8]

$$FAR = 0.1 \left(2 - \hat{\delta}^2_{mixture}\right) \frac{\rho_0}{\rho} \tag{7.2}$$

Engine static power is approximated as:

$$P_S = 1.093 \cdot 10^{-5} \left(1 + C_{\Delta P}\right) P_{max} p_{MAP} \left(\frac{n}{n_{max}} - 0.05\right) \tag{7.3}$$

Power losses can be calculated using following formula:

$$\Delta P = P_{max} C_{\Delta P} \left(\frac{n}{n_{max}}\right)^2 \tag{7.4}$$

Engine net power is given as:

$$P = P_S - \Delta P \tag{7.5}$$

7.2 Propeller

Thrust generated by the propeller and power required by the propeller are given by the following equations. [8, 20] Propeller revolution speed n is expressed in revolutions per second.

$$T = \rho n^2 D^4 C_T \tag{7.6}$$
$$P = \rho n^3 D^5 C_P \tag{7.7}$$

Where thrust C_T and power C_P coefficients are functions of advance ratio and blade angle.

Advance ratio is given by the following formula: [8, 20, 21]

$$J = \frac{V}{nD} \tag{7.8}$$

The propeller torque required is given as: [17]

$$Q = \frac{P}{2\pi n} \tag{7.9}$$

7.2.1 Propeller Induced Velocity

The pressure jump across the propeller disk can be expressed as:

$$\Delta p = \frac{1}{2}\rho \left[(V + V_i)^2 - V^2\right] \tag{7.10}$$

Hence:

$$T = \Delta p A \tag{7.11}$$
$$T = \frac{1}{2}\rho A \left[(V + V_i)^2 - V^2\right] \tag{7.12}$$

Induced velocity can be found by solving equation (7.12).

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